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Optimal Federal Public Debt Composition: Definition of a Long-Term Benchmark

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EXECUTIVE SUMMARY

Unit I: The Process of Federal Public Debt Strategic Planning

In Brazil, Federal Public Debt (FPD) strategic planning involves various aspects that can be grouped into three stages for didactic purposes:

- Definition of the desired long-term structure (benchmark);
- Medium-term planning (transition strategy); and
- Elaboration, publication and implementation of short-term strategy (Annual Borrowing Plan - ABP, first published in 2001).

The starting point and primary reference for the entire public debt planning process is a clear definition of its objectives. The objective defined for Federal Public Debt management is that of efficiently meeting federal government borrowing requirements at the lowest possible long-term financing cost, while ensuring prudent risk levels. Additionally, the aim is to the smooth operation of the Brazilian government securities market.

Considering the objective defined above as reference, the ABP presents a series of guidelines that orient the elaboration of FPD borrowing strategies. They are as follows:

- Increase the average maturity of the outstanding debt;
- Smooth the maturity profile, with special attention given to short-term maturities;
- Gradual replacement of floating-rate securities by fixed-rate and inflation-linked instruments;
- Improvement in the External Federal Public Debt (EFPD) profile through issuances of benchmark securities, buyback and structured operations;
- Development of the of the yield curve on both domestic and external markets and growth in the liquidity of federal government securities on the secondary market;
- Broadening of the investor base.

Taking into account the objective and guidelines set out in the ABP, macroeconomic scenarios, estimates of federal government borrowing requirements and the various public debt bond issuance strategies, each year the National Treasury calculates the expected values for the major FPD indicators: outstanding volume, composition by indexing factor, average maturity and percentage maturing in 12 months. On this basis, the ABP is used to announce the indicative limits of the upper and lower values that each of these indicators is expected to reach at the end of the year.

As the basis for ABP elaboration, strategic FPD planning defines a "transition strategy" from the current public debt composition to the long-term benchmark. The transition strategy seeks to respond to the following question: duly respecting the initial conditions (in other words, the current debt profile) and short and medium-term restrictions (particularly, macroeconomic restrictions and those implicit in the development of local financial markets), what should be the trajectory and speed of convergence to the desired long-term composition? The choice of transition strategies to the long term also explores the trade-offs between public debt costs and risks.

In its turn, the optimal long-term composition (benchmark) is the first stage to be discussed and approved by the Public Debt Committee, becoming the foundation for elaboration of the transition strategy and Annual Borrowing Plan approved for each year. In Brazil, development of the optimal public debt composition model was a natural consequence of a long process of improvement in the institutional framework used to evaluate FPD costs and risks. Initially, the government asset and liability management model was implemented. After that, the risk management instruments used by the National Treasury in FPD management were adopted. It was only at that point that studies were initiated on an optimal public debt composition model that would consider all of the relevant variables.

Unit II: The Analytical Framework of the Federal Public Debt Benchmark

With regard to defining an optimal long-term public debt composition (benchmark), it represents the desired profile for the debt structure and constitutes a guide for delineating the government's short and medium-term financing strategies. In the Brazilian case, the benchmark is expressed by a set of relevant debt indicators, including composition of the outstanding debt by type of index, average maturity and maturity structure, particularly the percentage of the debt maturing in the coming 12 months.

In defining the optimal public debt composition (benchmark), a set of models describes how relevant macroeconomic and financial variables to the public debt trajectory (interest rates, exchange rates, inflation and GDP) evolve over time. Based on simulated scenarios, evolution of the debt/GDP ratio is evaluated in order to derive cost and risk measurements of a given debt structure. Thus, following examination of multiple possible alternatives, one obtains the efficient frontier in terms of public debt costs and risks. On that basis, one chooses the structure in the frontier that possesses the desired profile for the long-term, complying with society's preferences between costs and risks.

An important question in the model is what should be the relevant concept of debt size in order to evaluate costs and risks. In the Brazilian case, we considered the Net Public Sector Debt/GDP ratio (NPSD/GDP) as the most relevant measurement, since it is the indicator most commonly utilized both by the federal government, to define its indebtedness targets and the primary surplus required to achieve them, and by analysts for the purpose of evaluating fiscal sustainability. Though the FPD is the National Treasury's work instrument, clear communication is sought between this debt and NPSD, the latter of which is a broader concept and an economic policy reference.

In the Brazilian case, the initial optimal composition proposal was published in the 2007 ABP. Simulations of this model suggested that efficient FPD management would result in growth in the proportion of fixed rate and inflation-linked securities, in detriment to the debt linked to floating interest rates or the exchange rate. More recently, refinement of the studies led to a definition of the desired composition, as presented in the 2011 ABP in the form of indicative long-term limits, as shown in the following table:

Optimal Long-term FPD Composition

	Lower limit	Upper limit
Fixed rate	40%	50%
Inflation-linked	30%	35%
Floating rate	10%	20%
Exchange rate	5%	10%

Source: National Treasury

The prescription that calls for seeking the composition described above should be qualified. First of all, it should be viewed as a guideline to be attained gradually, without generating pressures that could result in excessive transition costs. Secondly, one must avoid seeking FPD composition in any manner that is not coordinated with its maturity structure. Thirdly, the cost of altering the composition must be permanently monitored, since changes in the relative prices of public securities can result in adjustments in the FPD benchmark portfolio. Finally, though these limits provide up-to-date orientation for defining strategies, they must also reflect possible restrictions related to the development stage of Brazil's financial markets, the investor base profile and the outlook for future public security demand and liquidity.

UNIT I: THE PROCESS OF FEDERAL PUBLIC DEBT STRATEGIC PLANNING

Federal Public Debt – FPD strategic planning in Brazil involves various aspects that can be grouped into three stages:

- Definition of the desired long-term structure (benchmark structure);
- Medium-term planning (transition strategy); and
- Elaboration, publication and implementation of the short-term strategy (Annual Borrowing Plan – ABP, first published in 2001).

The objective of this unit is to describe the FPD planning process, highlighting Brazilian experience in debt risk evaluation[1].

This unit is organized into six sections aimed at clarifying the different elements of the planning process:

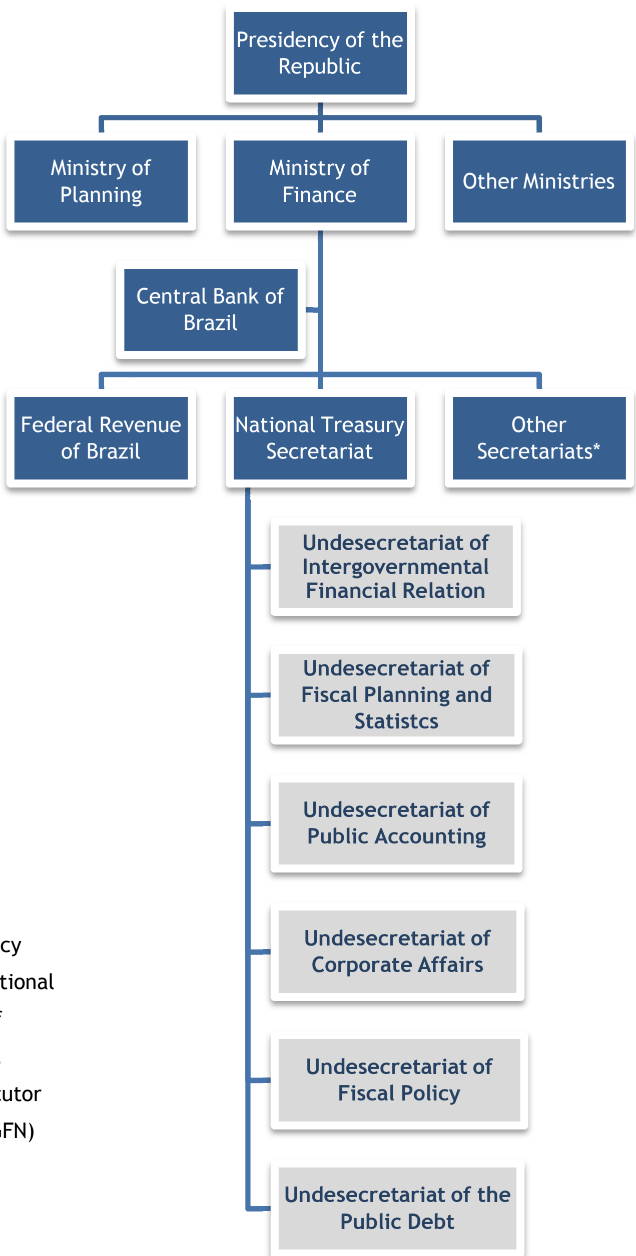
- Section 1 describes the institutional structure of the Public Debt Undersecretariat (SUDIP), charged with FPD management. Here, one should highlight the Debt Management Committee;
- Section 2 presents a broad explanation of the FPD strategic planning process;
- Section 3 breaks down the process of discussing and approving short-term FPD planning, set out in concrete form in the Annual Borrowing Plan;
- Section 4 presents a brief explanation of how the benchmark model aids in defining quantitative guidelines for the FPD structure over the long-term;
- Section 5 contains a discussion on how the strategy of transitioning from the current FPD composition to its benchmark is elaborated, with particular emphasis on the importance of the macroeconomic scenarios and the degree of debt market development to determining the speed of FPD convergence to the long-term structure;
- Finally, the final section details the historical evolution of the Brazilian strategic planning model, the risk indicators utilized for public debt management in the initial stages of this process and the more pertinent metrics currently used by debt managers.

[1] Based on SILVA, CARVALHO & MEDEIROS (2009) and also on the various Annual Borrowing Plans (ABP), released since 2001 by the National Treasury Secretariat .

1. Institutional Structure of the Public Debt Undersecretariat

The National Treasury Secretariat was created in 1986 with the objective of improving public finance management in Brazil. The National Treasury is a component of the institutional structure of the Ministry of Finance and is currently composed of six distinct undersecretariats (see figure 1 below). The first stage of this process occurred in 1988, when the functions of federal securities debt planning, supervision, regulation and control were transferred to the National Treasury. Previously, these functions had been scattered about various federal government organs and the Central Bank (BACEN).

Figure 1. Administrative Structure of the Federal



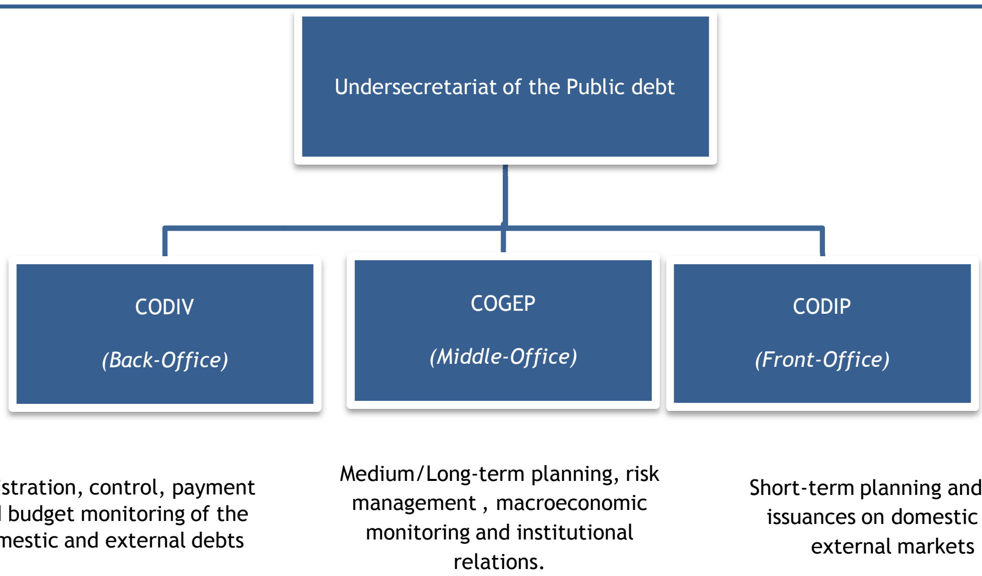
* Executive Secretariat (SE) Secretariat of Economic Policy (SPE), Secretariat of International Affairs (SAIN), Secretariat of Economic Monitoring (SEAE), Office of the General Prosecutor of the National Treasury (PGFN) and the College of Treasury Management (ESAF)

Source: National Treasury

The practice of concentrating public debt management functions in a single government entity has become widely accepted internationally, since this institutional arrangement tends to result in more efficient and coordinated management[2]. In line with international experience, the National Treasury is currently responsible for management of the entire Federal Public Debt (domestic and external debts, securities debt and contractual debt), a practice that has resulted in enhanced synergy for short and medium-term planning, operations in a diversity of markets, transparency and communication with different investor groups and risk classification agencies.

In this regard, in 1999 the National Treasury implemented a new management model based on the Debt Management Office – DMO philosophy. The Figure with the new National Treasury organizational structure is shown below. In this framework, the three SUDIP general coordination staffs participate in various segments of the public debt planning, issuance, registration, control and payment process: (i) Back-office (CODIV - General Coordination of Public Debt Control) is in charge of registration, control, payments and budget monitoring of the domestic and external debts; (ii) Middle-Office (COGEP - General Coordination of Public Debt Strategic Planning), responsible for medium/long-term planning, risk management, macroeconomic monitoring and institutional relations; and (iii) Front-Office (CODIP - General Coordination of Public Debt Operations), which is responsible for short-term planning and bond issuances on domestic and external markets.

Figure 2. Structure of the Public Debt Undersecretariat



Source: National Treasury

Within this institutional structure, FPD strategic management demands close coordination of proposed measures and monitoring of results during execution. In 2002, the Public Debt Management Committee

[2] In those countries in which public debt management responsibilities are distributed among various institutions, one can observe inconsistencies in processes and strategies, as well as duplication of functions.

was created and is composed of representatives of the three coordination staffs of the Undersecretariat of the Public Debt, together with the Undersecretary who presides over the Committee.

The Committee meets annually in order to analyze and approve the results of the optimal FPD composition model and to evaluate possible medium-term strategies for the debt. These discussions are the jumping off point for elaboration of the Annual Borrowing Plan, which is later discussed and agreed upon within the Committee before being sent to the National Treasury Secretary for final approval. The Annual Borrowing Plan is revised every four months in such a way as to evaluate whether changes in economic conditions may require significant alterations in initial planning.

The Committee meets in the final week of each month in order to define the short-term strategy to be adopted in order to achieve the targets set out in the Annual Borrowing Plan and to elaborate the official schedule of auctions in the coming month. These meetings are an opportunity for managers and analysts of the National Treasury Undersecretariat of the Public Debt to share their points of view and information regarding the current situation, markets and the future outlook for FPD financing.

Finally, one should stress that representatives of SUDIP's three coordination staffs, together with the National Treasury Secretary and Undersecretary, participate in the strategic planning process (including both short and long-term planning).

2. FPD Strategic Planning

The starting point for strategic planning is a clear definition of the objective of debt management. This objective may vary from one country to another. However, in general, it reflects an effort to attain adequate balancing between debt portfolio costs and risks and concerns regarding development of the government bond market.

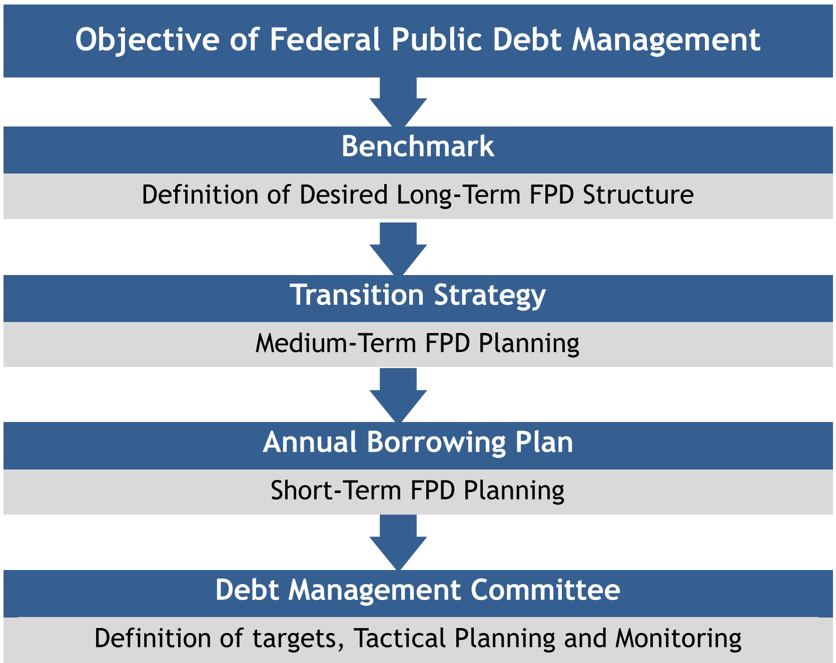
Objective of the FPD Management

The objective defined for Federal Public Debt management is that of efficiently meeting federal government borrowing requirements at the lowest possible long-term financing cost, while ensuring prudent risk levels Additionally, the aim is to the smooth operation of the Brazilian government securities market.

The objective described above is the starting point and major reference for the entire process of Brazilian public debt planning and management, as shown in Figure 1. This process will be discussed in detail in the following sections.

In summary, the process begins with the elaboration of analytical studies to be used as the foundation for discussion of the public debt benchmark choice. The next step is the design of a transition strategy that allows for more complete mapping of risks, opportunities and restrictions that may appear over the medium-term during the gradual shift toward the desired long-term debt portfolio. These elements are of critical importance to defining and developing the short-term strategy presented in the ABP, in terms of the tactical decisions taken by the Public Debt Management Committee.

Figure 3. Process of FPD Strategic Planning



Source: National Treasury

3. Annual Planning

Since 2001, the National Treasury has published an Annual Borrowing Plan (ABP) for the Federal Public Debt[3]. In the last decade, the ABP has become an instrument for strengthening the transparency and predictability of public debt management, while considerably improving National Treasury performance on

[3] Aside from the ABP, the National Treasury has also published the Annual Debt Report (ADR) since 2004. This Report presents the results achieved and the major events that marked public debt management in the previous year. Both the ABP and the ADR are available in Portuguese and English on the National Treasury website at: http://www.tesouro.gov.br/english/public_debt/index.asp.

the government bond market. The ABP is designed to serve the objective of Brazilian debt management and contains a series of guidelines that orient formulation of short-term strategy[4].

Aside from the FPD objective, these guidelines give due consideration to the benchmark and the transition strategy. In general, they encompass lengthening of the average maturity, smoothing of the maturity profile, increased participation of fixed rate and inflation-linked bonds, broadening of the investor base and government bond market liquidity and development of the term structure of interest rate.

FPD Management Guidelines

- Increase the average maturity of the outstanding debt
- Smooth the maturity structure, with particular attention to the short-term debt
- Gradual replacement of floating rate bonds by fixed rate or inflation-linked instruments
- Improvement in the external Federal Public Debt (EFPD) profile through issuances of benchmark maturities, buybacks and structured operations
- Development of the yield curve on both domestic and external markets and growth in the liquidity of federal government securities on the secondary market
- Broadening of the investor base.

Various alternative macroeconomic scenarios are considered during the annual planning elaboration. These scenarios are constructed on the basis of a set of relevant macroeconomic variables (short-term interest rate, exchange rate and inflation, mainly) and the hypothesis of preserving the main pillars of the economic policy adopted as of 1999 (inflation targeting system, floating exchange rate and robust primary surpluses).

Based on the estimate of Federal Government borrowing requirements for the year, the ABP guidelines, the various macroeconomic scenarios and different bond issuance strategies, the National Treasury calculates the values expected for the major Federal Public Debt – FPD indicators: amount outstanding, composition broken down by indexing factor, average maturity and percentage maturing in 12 months. At that point, the ABP presents the indicative upper and lower limits expected for each one of these indicators at the end of the year, as shown in Table 1.

[4]Although the current framework of FPD management calls for definition of the benchmark and design of a medium-term plan, its initial stages focus exclusively on the design of the short-term strategies that were set out in the annual borrowing plans

Table 1. Federal Public Debt Targets in 2011

Indicators	2010	Limits for 2011	
		Minimum	Maximum
Stock of FPD *(R$ billion)*			
	1,694.0	1,800.0	1,930.0
Profile (%)			
Fixed rate	36.6	36.0	40.0
Inflation-linked	26.6	26.0	29.0
Floating Rate	31.6	28.0	33.0
Exchange rate	5.1	4.0	6.0
Maturity Structure			
Average maturity (years)	3.5	3.5	3.7
% Maturing in 12 Months	23.9	21.0	25.0

Note: In the 2011 ABP, the National Treasury gathered under “Floating Rate” all bonds tied to interest rates subject to periodical resetting (e.g. Selic, TR and TJLP).
Source: National Treasury

Execution of the annual FPD planning demands rigid coordination between actions derived from planning and monitoring of results during the implementation stage. To avoid inconsistencies in ABP execution - and propose corrective measures, should that become necessary - the Debt Management Committee meets at the end of each month. During these meetings, questions related to the macroeconomic scenario are discussed, borrowing requirements and financial market conditions are evaluated, bond issuance strategy for the coming month is proposed and approved and its impacts in relation to the limits proposed in the ABP for the end of the year are assessed. The strategy includes such characteristics as the time to maturity and type of index (fixed-rate, floating rate, exchange variation or price index) of the financial instruments to be issued. This process also results in a definition of the public bond issuance schedule, specifying dates and types of auctions, coupled with the characteristics of the bonds to be issued.

4. Federal Public Debt Optimal Composition Model[5]

Definition of a financing strategy involves choices regarding the balancing of expected costs and risks that can be generated by a specific public debt structure. In this sense, based on social preferences between costs and risks, the debt manager must define the desired profile for long-term liabilities or, in other words, the benchmark, in such a way that financing of these liabilities can be done in the least burdensome manner possible without, however, resulting in increased risk exposure.

[5] See Unit II of this document for a detailed description of the model.

Brazil is not alone in the pursuit of a long-term public debt benchmark portfolio. Various countries, including Portugal, Sweden, Ireland, Denmark and South Africa, have used similar theoretical frameworks. Parallel to this, international organizations such as the World Bank and International Monetary Fund recommend that sovereign debt managers adopt benchmark models as a risk management and strategic planning tool.

In recent years, the National Treasury has developed a model to aid in evaluation of the cost and risk that different debt structures may generate for FPD, focusing specifically on the choice of its benchmark. The analytical framework of the model, which will be discussed in Unit II, is founded upon a set of simulations that make it possible to evaluate the behavior of alternative FPD portfolios in light of varied scenarios for the evolution of the variables that define the financing costs of the debt.

Based on these simulations, cost and risk indicators are derived for each portfolio evaluated. The lowest cost portfolio for a specific risk level is considered efficient. As the final product of the model, the bringing together of all of the portfolios that satisfied this condition for the different risk levels is denominated the efficient frontier, which expresses the trade-off in terms of costs and risks with which the debt manager has to cope. In the stochastic frontier, each portfolio is different in terms of composition, average maturity and percentage maturing in 12 months. The final results are the subject of debate among debt managers, giving due consideration to the feasibility of attaining the possible benchmark portfolios.

Finally, in choosing the benchmark, the public debt manager presents the results of the efficient frontier to the fiscal policy manager, who may be the Minister of Finance, Treasury Secretary or some type of Executive Committee. As the representative of society, the fiscal policy manager will be charged with choosing an acceptable risk level and, consequently, the borrowing cost desired by the government. In this process of choosing the benchmark, it is important to highlight that questions related to sustainability of the debt must also be given consideration. In this way, optimal compositions that have the potential for making the debt unsustainable, as a result of the projected financing cost or assumption of excessive risk, must be eliminated.

For a diversity of reasons, convergence of the current public debt composition to its benchmark may vary over time. Such factors as an adverse macroeconomic environment or the absence of a developed local debt market - which could result in shortfalls in demand for some preferred debt instruments - may reduce the speed of the convergence process. Under such circumstances, more robust theoretical analysis, simulations of the dynamics of the debt in alternative scenarios and deepening of debates on the definition of long-term objectives may be as or even more important to debt management than concentrating efforts on the identification of possible optimal compositions.

The initial proposal of the Brazilian optimal composition model was published in the 2007 ABP. Simulations of this model indicated that efficient FPD management would be that which resulted in greater participation of fixed-rate bonds[6] and inflation-linked bonds, in detriment to floating rate or exchange-

[6] Among the advantages of fixed-rate bonds, mention should be made of the following: (i) they ensure greater predictability regarding debt costs; and (ii) they contribute to development of the country's fixed income market.

rate-indexed bonds[7]. Following the guidelines above, the recent evolution of the FPD profile has resulted in greater equilibrium between FPD costs and risks.

Annually, as shown in Table 2, discussions have moved forward to refining of the long-term FPD quantitative guidelines, including definition of indicative limits to be sought in the period. Although these limits provide guidance for defining strategies, it is important to emphasize that they also reflect possible constraints related to the Brazilian macroeconomic scenario and the development stage of local financial markets. The speed of convergence from the current FPD composition to that indicated in Table 3 will depend on surmounting some of these constraints.

Table 2. Indicative Intervals of the Desired FPD Composition over the Long-Term

	Lower limit	Upper limit
Fixed rate	40%	50%
Inflation-linked	30%	35%
Floating rate	10%	20%
Exchange rate	5%	10%

Source: National Treasury

5. Elaboration of Medium-Term Transition Strategy

Aside from informing society with regard to the short-term (one year) strategy and optimal long-term composition (benchmark) through publication of the ABP, strategic FPD planning annually defines a transition strategy from the current public debt composition to the long-term benchmark. The transition strategy seeks to respond to the following question: duly respecting initial conditions (the current debt profile) and short and medium-term restrictions (particularly, macroeconomic restrictions and development of local financial markets), what should be the convergence trajectory and speed toward the desired long-term composition?

Development of the transition strategy demands elaboration and discussion of qualitative and quantitative macroeconomic scenarios for the variables (mainly, the Selic rate, exchange rate, inflation and GDP) that impact FPD costs and risks. In each of the proposed scenarios, different strategies are evaluated in light of

[7] As regards the debt tied to exchange rate (currently restricted to the external debt), simulations demonstrate that, though this debt tends to have lower average costs, its risk level is extremely high, particularly in stress scenarios. Despite this and in light of the protection provided by international reserves in the case of exchange rate fluctuations, taking some degree of exchange rate risk in the FPD can be considered interesting from the point of view of the Net Public Sector Debt (NPSD). At the same time, sovereign bonds are important to the development of the Brazilian yield curve on the international market, and serve as a reference for the private sector in Brazil.

the speed of convergence to the long-term benchmark. In practical terms and respecting the borrowing conditions set down for each scenario during the transition period, the speed of convergence toward the benchmark is the main difference among the alternative strategies.

The choice of transition strategies for the long-term also takes advantage of the trade-offs between public debt costs and risks. The results for the cost, risk, maturity profile and debt composition indicators are simulated for each strategy. For example, a public debt manager could choose to limit issuances essentially to fixed-rate debt, while another may attribute greater importance to issuance of inflation-linked bonds. Alternative choices such as these aid data managers in the decision-making process, since they show the consequences of their choices for the major debt indicators.

In conclusion, elaboration of the transition strategy involves integration of the processes of benchmark simulation and definition of the convergence strategy. While the benchmark model is based on the assumption that the economy is in a steady-state[8], the transition strategy is defined on the basis of possible scenarios for the coming years. For this reason, discussions on macroeconomic scenarios and evolution of public debt markets are vital to determining how the economy will converge to its stationary state and, therefore, how public debt management must be implemented in order to attain the benchmark[9].

6. Risk Evaluation: Brazilian Experience

Evaluation of risk exposure is an important element of the planning process and one of the foundations for defining FPD strategic guidelines[10]. Integration of risk management tools into FPD risk evaluation instruments must be viewed together with the economic evolution of the country, which has favored development of the public debt markets, thus expanding the array of financing alternatives. Economic stability and market improvements were conditions of essential importance to the evolution of public debt planning and analysis instruments.

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[8] In the public debt benchmark model, the steady state has two meanings. In the first place, it means that all of the economic variables are fluctuating around their long-term equilibrium values. In practical terms, the stationary state scenario encompasses the following characteristics: stability of the economic environment, reduced fiscal vulnerability, lower interest rates, controlled inflation and sustainable economic expansion. The second meaning found in the idea of a steady state is that each debt composition is associated to an issuance strategy that maintains constant the long-term FPD portfolio characteristics.

[9] The alternatives for the transition strategy are simulated in a deterministic context. It is possible to design dynamic stochastic simulation systems with the aim of also optimizing the transition strategy. This approach is expected to be the next step in application of optimization models. However, since it is still incipient and highly complex, we have been unable to find any sovereign debt manager who has successfully used it.

[10] More detailed information can be encountered in ALVES & SILVA (2009) and SILVA, CABRAL & BAGHDASSARIAN (2009).

The major risks involved in FPD management are refinancing and market risks, though due attention should also be given to strategic, operational and legal risks. A more detailed description of these types of risk is found in the following chart.

Major Risks Monitored in FPD Management

Refinancing Risk

• **Refinancing risk** is based on the possibility of having to cope with higher costs in order to obtain short-term financing or, in an extreme situation, of being unable to refinance the debt maturing over the short-term. This risk is related to the debt maturity profile, as well as to its short-term sensitivity to shocks in the major variables..

Maket Risk

•**Market risk (or financial risk)** is derived from variations in the financing costs resulting from movements in the short-term interest rates, the yield curve, the exchange or inflation. In other words, this risk is related to fluctuations in the debt stock on the market.

Strategic Risk

•**Strategic risk** results from the possibility of a strategy not achieving its objectives. For the national Treasury, this risk is present, for example, in the possibility of not achieving the limits set down in the ABP for FPD composition by indexing factor, due to the choice of an inadequate strategy.

Operational Risk

• The Concept of **Operational Risk** is quite broad and encompasses the possibility of failures caused by persons, internal processes or systems or, furthermore, external events that generate losses for institutions. In the case of the National Treasury, this risk may be perceived by the external public, mainly through government bond auctions. For example, an interruption in electricity transmission could make it unfeasible to sell bonds at the programmed moment of the auction.

Legal Risk

•**Legal Risk** results from the possibility of not respecting the limits for debt indicators as expressed in legislation (annual volume of issuances, for example).

In Brazil, the development of the public debt optimal composition model was a natural consequence of a long process of improvement in the institutional framework it utilized to evaluate FPD costs and risks. Initially, the government's asset and liability management model was implemented. Following that, the risk management instruments used by the National Treasury were adopted in FPD management. At that point, studies were initiated on use of the optimal composition model for the public debt.

To facilitate understanding of this evolution over time, one can divide it into three different stages. In the first stage, the National Treasury adopted guidelines for FPD management using the Asset and Liability Management - ALM model as the basic reference. The objective of this model is to combine the characteristics of government assets and liabilities in such a way as to protect the net debt against market risks and thereby smooth out oscillations in the government asset balance. With this in mind, ALM considers the public debt management strategy and other macroeconomic policies. The 2002 ABP[11] was the first report to mention the ALM model.

The result of the implementation of the ALM model by the National Treasury was the elaboration of periodical monitoring reports on the assets and liabilities for which the Central Government was responsible. These reports made it possible to develop more effective financing strategies in terms of the balance between these assets and liabilities. The reports in question identified the mismatches between assets and liabilities in terms of indexing factors, average maturities, cash flows and the percentage of debt in the following 12 months, and included simulations of the future evolution of these mismatches.

The objective of the ALM model is to create a portfolio of liabilities with risk characteristics similar to those of the government's assets and, in this way, reduce the sensitivity of the public sector asset balance to shocks in economic and financial variables. Considering the particularities of the public sector, it is important to adequately map the assets to be included in the balance used for ALM purposes. The fact that the government does not seek to maximize profits and is able to charge taxes results in a situation in which the ALM structure differs from the public to the private sector.

Normally, one starts with an accounting balance, which is adapted to the "economic balance", which includes only those items that represent potential financial liabilities and those that will contribute to paying them. In other words, one should consider only assets/liabilities that interfere in sovereign risk. This rule may result in exclusion of assets that did not generate financial flows for debt management purposes and inclusion of contingent liabilities in the government balance. On the other hand, illiquid assets such as national parks, military equipment, government buildings are included in the accounting balance, but may be considered irrelevant to analysis in the framework of the government's ALM. However, should these items be incorporated into a privatization program, they would be included in the asset map.

Finally, one should stress that the major asset of a government is its capacity to collect taxes. In contrast to this, there are certain expenditures that are typically the responsibility of the public sector. From the ALM point of view, the debt manager must be capable of perceiving the characteristics of revenues or of

[11] http://www.tesouro.gov.br/english/public_debt/downloads/paf2002eng.pdf.

future primary surpluses and, whenever possible, estimate them with the objective of "matching" the balance sheet. Over the long-term, the current value of the debt must be financed by the present value of the sum total of future primary surplus flows.

In the second stage of the process of improving the FPD risk management mechanisms, approaches were aggregated seeking to measure the impact of adverse shocks in debt indexing factors on FPD. This was the case of the stress test, which simulates the negative impact on the outstanding volume (or cost) of FPD as a result of strong and persistent pressure on the real interest rate or on the exchange rate. Another example was the initial use of stochastic indicators[12], such as Cash-flow-at-risk (Cfar)[13] and Cost-at-risk (CaR)[14]. These indicators are obtained on the basis of simulations utilizing the Monte Carlo method for key variables (interest, exchange and inflation rates, in the Brazilian case), calibrated by historical data or deterministic parameters. As a result of these simulations, probability distributions of cash flow (CfaR) or outstanding debt (CaR) are obtained. This class of indicators made it possible to estimate losses expected in the debt as a result of negative events and were initially presented in the 2004 ABP[15].

Table 3 below presents a brief description of the major indicators used to monitor FPD risks in Brazil.

[12] Risk indicators based on stochastic simulations that have the advantage of indicating a probability distribution of the value of payment flows or the value of outstanding debt. This type of risk measurement makes it possible to estimate losses for the public debt consequent upon negative events that may occur in the economy, aside from quantifying the probability of such events.

[13] Cash-flow-at-risk indicates the maximum increase that may occur in FPD payments flows for a given period, in relation to the expected value of such payments, with a given probability (e.g. 95% confidence level).

[14] In Brazil, Cost-at-risk is used to measure uncertainty in relation to the volume of debt at the end of a period. It indicates the maximum value that the debt can reach, for a given probability. Differently from the case of Denmark, in which the measurement is defined in terms of debt costs, the Brazilian CaR is determined on the basis of outstanding debt. However, the two approaches are equivalent, since, the greater the cost, the greater will be the outstanding debt for a given government primary result.

[15] http://www.tesouro.gov.br/english/public_debt/downloads/PAF_2004_ingles.pdf.

Table 3. Major public debt risk indicators used in Brazil

Type of Risk	Indicator	Description	Comments
Refinancing risk	Percentage of FPD maturing in 12 months	Indicates the short-term concentration of debt maturities.	A complete description of the debt maturity profile can be drawn up on the basis of an analysis of the entire FPD maturity structure (that is, also evaluating the percentage of debt maturing over the medium and long-term)
	Average FPD maturity	Indicates the average period of time in which the debt should be paid or refinanced.	In Brazil, the resent value of debt flows (principal and interest) is used as the weighting factor of the terms of each flow (concept of duration) in calculating the average FPD maturity.
	Cash-Flow-at-Risk (CfaR)	Indicates the maximum increase that may occur in FPD payment flows projected for a given period, considering a given confidence interval (e.g. 95% probability)	CfaR and CaR (see below) are risk indicators based on stochastic simulations, with the advantage of indicating a distribution of probabilities of the value of payment flows or of the value of the outstanding debt. This type of risk measurement makes it possible to estimate public debt losses consequent upon negative events that occur in the economy, as well as to quantify the probability of such events.
Market Risk	FPD Composition	Indicates the percentage of the outstanding debt by type of index	The types of index that categorize outstanding FPD in this indicator are defined according to classes of risk which, in turn, depend on debt indexing factors. In the case of FPD, there are 4 classes: Fixed Rate, Floating rate, Inflation-Linked (Price Indices), Exchange-rate
	Refixing Risk (or interest rate risk)	Indicates the share of FPD subject to cost increases caused by short-term interest rate fluctuations.	Corresponds to the debt exposed to interest rate fluctuations, caused by the fact that the debt must be refinanced (at new rates) or because earnings on the debt are generated by floating rate(e.g. Selic rate). Thus, the indicator is given by the sum total of percentages of FPD maturing over 12 months and the percentage with floating rate maturing after 12 months.
	FPD Sensitivity Analysis	Indicates the increase in the outstanding debt (cost) as a result of a 1% variation in a specified indexing factor (short-term interest rate or exchange rate).	This analysis seeks to respond to the following question: "What happens if a specific shock occurs?". Alternatively, this indicator may be calculated by assuming a variation equivalent to one standard deviation in the benchmark indexing factor.
	Stress Test	Measures the negative impact on outstanding FPD (or on its cost) due to strong and persistent pressure on real interest rate or real exchange rate.	This is equivalent to the sensitivity analysis. However, in this case, one applies a shock equivalent to 3 standard deviations of the real interest rate or real exchange devaluation accumulated over 12 months to outstanding FPD.
	Cost-at-Risk (CaR)	Indicates the maximum value that outstanding FPD may reach at the end of a specified period of time (e.g. 1 year) for a given confidence interval (e.g. 95% probability).	In Brazil, cost-at-risk is used to measure uncertainty with respect to the volume of debt at the end of the period. Despite being defined in terms of outstanding volume, instead of cost (interest), these two approaches are directly related, since, the greater the cost, the greater will be the debt stock for a given primary government result.

Source: National Treasury

Having defined risk management instruments, the final stage of institutional development resulted in efforts to elaborate a model that would further refine general FPD guidelines quantitatively. In other words, definition of an optimal long-term composition model for FPD was sought, with the purpose of minimizing impacts of public debt shocks on the fiscal result. The initial proposal of the model was published in the 2007 Annual Borrowing Plan[16] and later in CABRAL et. alli. (2008). A more detailed description of the analytical framework used by the National Treasury to aid in defining the FPD benchmark will be discussed in Unit II of this document.

In conclusion, though utilization of sophisticated financial instruments in FPD management has transformed the National Treasury of Brazil into a reference for like institutions in the world, there is still room for improvement in the model. By way of example, one could include utilization of macrostructural models for generation of scenarios, in which macroeconomic and financial models are conjugated in the generation of stochastic scenarios, and use of different approaches to modeling of the yield curves[17], inflation and exchange rates.

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[16] http://www.tesouro.gov.br/english/hp/downloads/annual_borrowing_plan2007.pdf.

[17] Currently, a COX, INGERSOLL & ROSS (1985) - based model is utilized, known as CIR models, with just one factor (interest rate level) to explain the forward interest rate structure (ETTJ). In the future, one expects to work with models that include the dynamics of more yield curve factors (for example, level and inclination), whether they be derived from the CIR family or based on other specifications such as those derived from NELSON & SIEGEL (1987).

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UNIT II: THE ANALYTICAL FRAMEWORK OF THE FEDERAL PUBLIC DEBT BENCHMARK

As we saw in Unit I, the definition of the optimal public debt composition is one element of the strategic planning process. Optimal composition (benchmark) represents the profile desired for the long-term debt structure and acts as a guide for elaborating the government's short and medium-term financing strategies.

In the Brazilian case, the benchmark is expressed by a set of indicators of relevance to the debt, including composition of outstanding debt by type of index, maturity structure, particularly the percentage of debt to mature in the next 12 months, and the average maturity of the outstanding debt. Implementation of the benchmark can be achieved through definition of targets for the values to be attained by these indicators over a specific temporal horizon.

This Unit is an effort to describe the model utilized by the National Treasury in evaluating the trade-offs between costs and risks derived from alternative profiles for the Federal Public Debt (FPD) structure over the long-term, based on the objectives and guidelines defined for management of that debt[18].

- This unit is organized into four sections:
- Initially, we will present the main theoretical arguments in favor of the adoption of a benchmark, together with information on international experience in this area;
- In section 2, we will describe the simulation model used by the National Treasury;
- Application of the model for defining the FPD benchmark is illustrated in section 3;
- Finally, section 4 presents FPD composition desired for the long-term, in the form of upper and lower limits.

[18] Definition of a FPD benchmark was discussed in such previous studies as CABRAL & LOPES (2005), SILVA, CABRAL & BACHDASSARIAN (2006), CABRAL et alli. (2008) and ALVES (2009).

1. Literature and International Experience

The importance of the optimal composition (benchmark) is clearly supported by theoretical literature, which stresses the relevance of public debt management to economic activity[19]. This is particularly true regarding the literature on tax smoothing and temporal consistency, which supports active debt management. The theoretical arguments in favor of seeking an adequate debt composition take on even greater relevance when one considers the elements discussed in literature on the credibility of macroeconomic policies, signaling and real effects of a sovereign default, among others[20].

In this debate, one should also mention the contribution made by multilateral institutions, including the World Bank and International Monetary Fund. In their publication Guidelines for Public Debt Management (WB; IMF, 2001), these two institutions describe the benchmark as a powerful tool for representing the debt profile that the government desires to attain, based on its preferences as defined by the trade-off between costs and risks.

Finally, international experience has documented that various countries have taken measures to define an optimal composition for their debts. A case in point is Portugal, one of the pioneers in the formulation and adoption of optimal long-term composition for quantifying the objective of its public debt management, as well as for enhancing the consistency between daily decisions and the long-term objective. Denmark, Sweden, Canada and the United Kingdom have also developed models designed to aid in defining a benchmark portfolio to be used as a guide in elaboration of financing strategies. The following chart summarizes aspects of the experience of these countries.

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[19] The Hypothesis of Ricardian Equivalence is an important point of departure in this debate, though it does not provide subsidiary information for an active defense of public indebtedness. One consequence of the Hypothesis of Ricardian Equivalence is the neutrality of the debt with respect to economic activity, since debt and taxes are equivalent from the intertemporal point of view. However, Ricardian Equivalence is basically founded upon the following suppositions: 1) an infinite planning horizon; 2) complete markets; and 3) taxes that do not cause distortions (BARRO, 1974; 1979; 1989). With relaxation of these presuppositions, new theories have drawn conclusions regarding the importance of adequate debt management.

[20] See GOLDFAJN & DE PAULA (1999).

Tabela 1. International Experience

Country	Relevant indicator	Summary of the model	Source
Portugal	Cost and risk of cash flows, with restrictions regarding refinancing risk	The model assumes a constant nominal debt in a stationary state and has three inputs: 1) stochastic interest rates; s) financing strategies; and 3) deterministic scenarios for other macroeconomic variables. A subset of the best solutions of the model is presented to the authorities charged with the final decision.	IGCP (1999).
Sweden	Targets for the participation of each type of debt, based on cash flows and duration	Running Yield: probability distribution of the model is calculated through dynamic simulation of the interest (short and long-term), inflation, exchange rate and GDP curves. A measurement of dispersion of this distribution is used as risk indicator.	RIKSGÄLDEN (2008); RIKSGÄLDEN (2009)
Ireland	Current net value and fiscal volatility	The current net value is used as a cost measurement, and fiscal volatility as a risk measurement. Since it reflects the structural conditions of the economy and the final objective of fiscal policy, the benchmark should not be altered significantly over time. In this sense, revisions in the benchmark can be implemented to reflect structural changes in the economy, but not in response to short-term movements.	NTMA (2006); NTMA (2011)
Denmark	Duration of the portfolio	In order to define the duration target, a long-term analysis of the evolution of expected costs is carried out. The model considers only the domestic debt. Interest rate risk is dealt with in the ALM approach and the trade-off between costs and risks is evaluated by a Cost-at-Risk model. The model combines stochastic and deterministic scenarios.	DANMARKS NATIONALBANK (2007)
England	Debt Service Cost (in terms of cash flows) as a proportion of GDP	The benchmark is not utilized. However, models are utilized to illustrate the impact of different issuance strategies and indicators for risk management. Long-term analysis of financing strategies is done through the use of stochastic scenarios based on the combination of a macroeconomic model with specifications for interest curves. Cost is measured by cash flows and risk by the dispersion of payments.	UK-DMO (2011) e PICK AND ANTHONY (2006)
Canada	Cost measurement: annual average debt service burden as a percentage of total outstanding debt Measurement of risk: cost of volatility or consideration of impact on the budget	Combines a macroeconomic model with interest curves to simulate costs and risks of alternative financing strategies. In order to aggregate more than one objective into the analysis, the model offers a tool for minimizing the weight of the debt service with restrictions on other objectives.	BOLDER (2008)

Elaboration: National Treasury

2. The Brazilian Optimal Composition Model

2.1. Some Methodological Issues

The analytical framework utilized in the study of optimal FPD composition is based on stochastic simulations derived from finance and efficient portfolio theories. However, before proceeding to a description of the model itself, some comments must be made regarding the direct application of the instruments of traditional financial analysis to government policies.

In general, the government, with the condition of preserving prudent risk levels, may have more complex objectives than reducing costs. Aside from this, evolution of its cash flows and indicators of budget impacts may have implications for the choice of the optimal debt structure. One should also consider that, given the nature of the public debt, government measures have a strong impact on bond prices and, consequently, on the cost and risk of its financing strategies. As a result, these peculiarities may lead economic policy managers to define a debt composition as benchmark that is different from those on the efficient frontier, obtained from the strictly financial point of view.

One important question in the model refers to what should be the relevant debt concept for evaluating costs and risks. In the Brazilian case, the National Treasury only has direct control over the Federal Public Debt, which encompasses all domestically and internationally issued bonds, as well as the federal government's external contractual debt. Nonetheless, the most commonly used indicator, both by the federal government, to define its debt targets and the primary surplus required to achieve those targets, and by analysts, with the aim of evaluating fiscal sustainability, is the ratio between the Net Public Sector Debt and GDP (NPSD/GDP). This concept is more inclusive since it encompasses all public sector liabilities, deducted from its assets against other economic agents. Public sector is understood as the federal government (including the Social Security System), Central Bank, state and municipal governments and public sector companies.

The reduction in NPSD volatility (risk) is important to the extent in which the occurrence of shocks with potential to jeopardize its sustainability requires a fiscal policy response. In this sense, unforeseen NPSD fluctuations may result in tax surprises that would affect the available income of the population, generating inefficiencies from the social welfare point of view[21]. Despite the fact that the National Treasury's work instrument is the FPD, there is clear communication between this debt and the NPSD, which is much broader and used as an economic policy reference.

For the reasons set out above, a decision was made to use the NPSD/GDP indicator for choosing the optimal composition in Brazil. This choice was based on the idea that, in an analysis of government intertemporal budget restrictions aimed at evaluating the sustainability of the public debt, all public sector assets and liabilities should be considered. As a matter of fact, many economic analysts and

[21] Literature on optimal taxation suggests that if taxes cause dead weight losses, the government should smooth them over time, thus minimizing distortions consequent upon revenue inflows. In this case, the debt profile and its risks are relevant for public policy purposes, since fluctuations in debt costs would result in alterations in the tax load. See BOHN (1990).

financial market participants, including international organizations (e.g., the World Bank and IMF) and rating agencies (e.g., Standard & Poor's), consider NPSD/GDP as the relevant indicator for evaluating the sustainability of the Brazilian debt.

Another important aspect refers to the fact that the benchmark study is based on the premises of the steady-state. This has two meanings for the model. In the first place, it presupposes that the economy is already in a steady state or, in other words, that all variables are fluctuating around their long-term equilibrium values. The fact of the matter is that this supposition is appropriate to a discussion of a debt profile desired for the long-term, avoiding the possibility of the decision being contaminated by transitory fluctuations in economic scenarios. In practical terms, the reference to the steady state scenario includes the following characteristics: stability of the economic environment, low fiscal vulnerability, real interest rates already in a state of equilibrium, controlled inflation and sustainable economic growth.

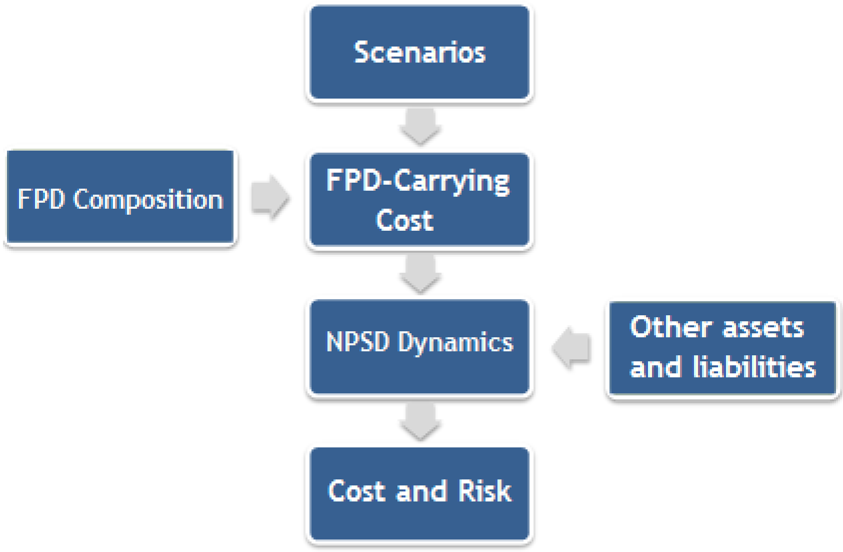
The second meaning found in the idea of a steady-state is that each issuance strategy implicitly preserves constant the characteristics in the long-term FPD portfolio. In other words, the strategy itself and, therefore, debt management guidelines must be stable over time, with no sharp fluctuations caused by temporary shocks in the economy, coupled with avoidance of nearsighted behavior guided by short-term parameters.

2.2. The Optimal Composition Model

The study of the optimal composition (benchmark) for the Brazilian debt is based upon the application of stochastic simulation methods, with the objective of deriving an efficient frontier of debt compositions, expressing potential trade-offs between costs and risks in FPD management. In this sense, a composition is viewed as efficient when it has the lowest risk for a specified cost level or, alternatively, when the cost is the lowest for a specified risk level. The entire set of compositions that satisfied this condition defines the efficient frontier, while it is the debt manager's task to choose which composition is desirable, since it is not possible to obtain simultaneous cost and risk reductions among the frontier portfolios.

The following Figure illustrates the general idea of the model used for analysis of the trade-off between costs and risks.

Figure 1. Schematic Summary of the Model



Source: National Treasury

Initially, various stochastic scenarios are generated for the principal macroeconomic and financial variables – output, inflation, exchange rates, short-term interest rates and public bond prices – with the objective of simulating evolution of the major factors that influence the trajectory and cost of the public debt over time.

At that point, a debt composition is chosen that involves a basket of securities, with varied types of indexes and different maturities. By way of example, we will look at a portfolio composed equally of one year fixed rate bonds and five-year floating rate bonds. Over the course of each simulated scenario, the FPD financing cost is calculated for the chosen debt composition.

The next step of the simulation process involves calculation of FPD and NPSD evolution which depends on the previously obtained FPD financing cost, as well as on other parameters that define the public sector primary result and evolution of "other assets and liabilities[22]" that make up the NPSD.

Finally, cost and risk indicators are derived from analysis of the behavior of NPSD in the face of stochastic shocks. Since thousands of scenarios are simulated and the value of the NPSD is calculated for each one of

[22] FPD includes only the domestic and external debts for which the National Treasury is liable. In calculating NPSD, the definition of public sector used to measure indebtedness is that of the nonfinancial public sector plus the Central Bank. Consequently, it encompasses the direct federal, state and municipal administrations, indirect administrations, the public social security system, nonfinancial federal, state and municipal government companies, as well as the Central Bank of Brazil. In this concept, intragovernmental debts are excluded, so as to measure only the public sector debt with private agents. Since NPSD is a net debt concept, public sector liabilities are deducted from its assets with other economic agents.

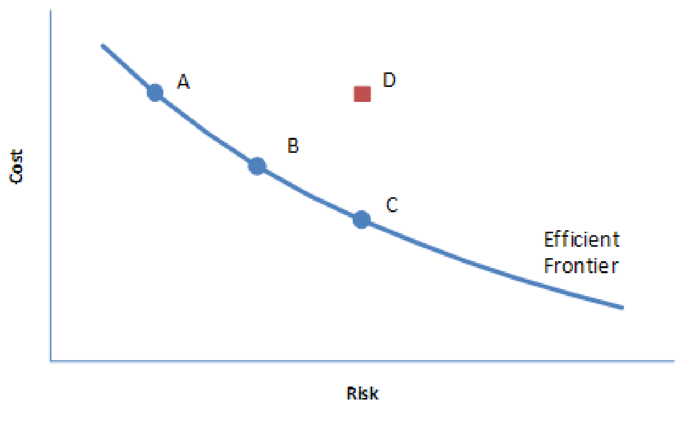
them for a given point in time, in practical terms the results provide a distribution of probabilities of the value of NPSD, from which the costs and risk metrics are extracted.

In the simulations, the initial FPD structure, stochastic scenarios, primary result parameters and "other assets and liabilities" are the same for any debt composition chosen. As a result, the only reason for the resulting trajectory of NPSD simulations to vary among different debt compositions is the selected composition itself.

After performing the simulations, the metrics obtained for each evaluated composition are plotted on a graph in which the axes are NPSD/GDP cost and risk, in such a way that the efficient frontier is obtained as a curve composed of the points that represent the lowest cost for a specified risk level. The frontier portfolios are efficient because it is not possible to alternate among these portfolios in order to obtain cost and risk reduction gains simultaneously. Finally, given the government's position on risk (which should reflect that of society), it is possible to choose a specific frontier portfolio that will define the debt benchmark.

The following figure illustrates this concept. Portfolios over the length of the frontier (A, B and C) are efficient because the risk necessarily increases when one seeks to reduce the cost of the debt alternating between these compositions (from A to B; or from B to C). Compositions above and to the right of the frontier are inefficient because they increase risk, given the level of costs (D compared to A), or increase the cost for a given risk level (D compared to C), or increase both the cost and the risk, in comparison to an efficient composition (D compared to B).

Figure 2. Illustration of the Efficient Frontier



Source: National Treasury

In operational terms, the efficient frontier depends on the average cost and standard deviation of each bond, just as occurs with the cost correlation matrix of these securities. In the first place, simulations based on debt compositions with 100% of a specified bond provide the average cost and standard deviation. Secondly, simulations based on portfolios with 50%-50% pairs of bonds aid in calculating the cost correlation matrix[23,24]. Consequently, with these data in hand, a numerical procedure is employed in order to discover the lowest cost portfolio for each possible risk level, concluding with the obtaining of the analytical efficient frontier.

In the case of the FPD benchmark, portfolios can be composed of four basic instruments which differ with respect to their characteristics: fixed rate bonds, floating rate bonds, inflation-linked bonds and bonds denominated in foreign currencies. Each one of these categories also differs in terms of maturities, thus making it possible to specify a basket of representative short, medium and long-term bonds. More specifically, the financing instruments considered are as follows:

- Fixed rate : 1, 3, 5 and 10 years;
- Floating-rate (indexed to the Selic rate): 5 years;
- Inflation-linked : 10 and 30 years;
- Denominated in foreign currency (exchange rate): 10 and 30 years.

Although the bond listing above seeks to reflect the financing options currently available for FPD financing, the simulation framework has the flexibility needed for inclusion of other securities, currencies and maturities. Besides this, in generating the efficient frontier it is possible to include such technical restrictions as a minimum (or maximum) percentage for participation of a security or category in the debt portfolio, or a minimum average maturity for the optimal portfolio.

Next, two large sets of procedures used to simulate the model are emphasized. In the first place, the generation of scenarios for simulating the dynamics of the economy and calculating the cost of debt financing are shown. Right after that, the dynamics of the debt from which the cost and risk indicators result in the model will be presented.

2.3. Dynamics of the Economy

The set of simulations depends on the generation of scenarios for the economic variables that determine the cost of debt financing and the dynamics of the NPSD/GDP ratio. With this, the model requires specification of a set of equations used to describe how these variables evolve over time.

[23] The correlation between the cost of two bonds is extracted through the following ratio: cost variance (σ_p^2) of a portfolio with two securities is $\sigma_p^2 = w_1^2\sigma_1^2 + (1 - w_1)^2\sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2$, in which σ_1 is the cost variance of security 1; σ_2 is the cost variance of security 2; ρ_{12} is the correlation between the costs of securities 1 and 2; $0 \leq w_1 \leq 1$ is the relative weight of security 1 in the portfolio.

[24] The correlation matrix must be semi-defined positive. When this does not occur, a spectral decomposition is utilized to obtain a semi-defined positive matrix that is close to the original. This decomposition considers that when a matrix is not semi-defined positive, it has at least one negative eigenvalue. The procedure for decomposition utilizes the positive eigenvalues of the original matrix and substitutes the negative values for zero in order to recompose the matrix. This method provides a reasonable approximation for the correlation matrix. The steps for spectral decomposition can be seen in JÄCKEL (2002).

In this sense, the basic processes of the model cover the following variables:

- Basic interest rate (Selic);
- Term structure of interest rates for:;
    - fixed-rate securities;
    - IPCA-linked securities;
    - securities denominated in foreign currency;
- Inflation rates (domestic and foreign);
- Exchange rate (real and nominal[25]);
- Gross Domestic Product (GDP).

Aside from the deterministic component, the equations that describe the evolution of the variables above add a stochastic term with the objective of simulating random shocks in their trajectories. One assumes that these stochastic shocks follow a correlation structure, thus conferring macroeconomic consistency on the simulations.

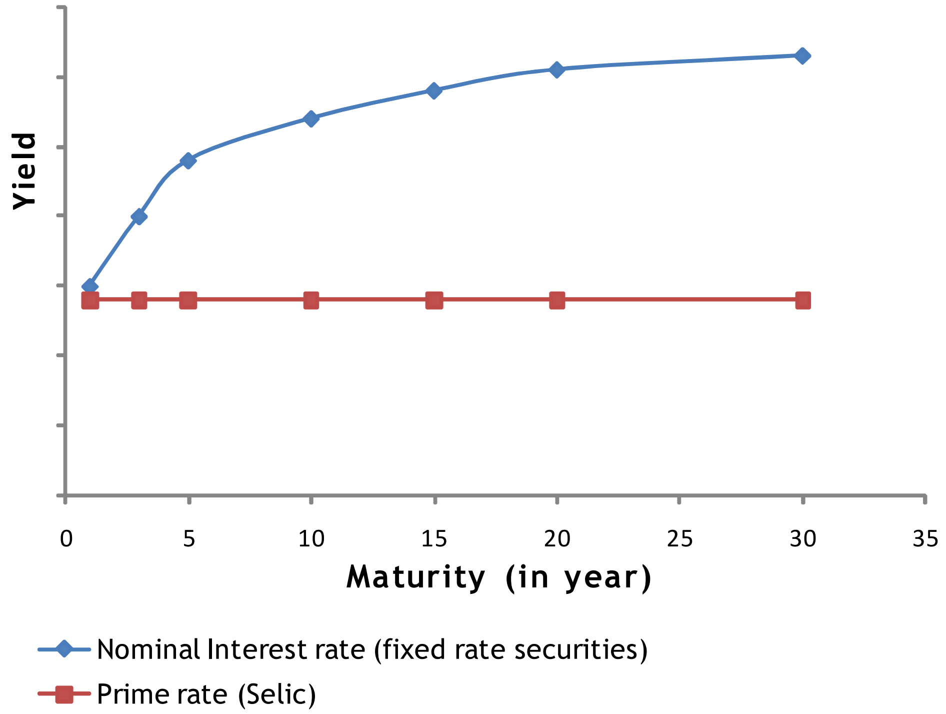
Particularly in relation to the yield curves, the scenarios are necessary in order to obtain the cost of each financing option. Although the cost of FPD financing depends primarily on the prime interest rate of the economy, each debt instrument has particularities, especially with regard to indexing factors, maturity and degree of liquidity.

The most basic case applies to floating rate bonds (Selic rate). The model presumes that they are sold at par or, in other words, their price is equal to face value. The cost is then defined by the Selic rate composed on a daily basis throughout the period, independently of its maturity.

Given the other alternatives available for FPD financing, the model has specific yield curves for each type of instrument. Although cost is based on the short-term interest rate, each debt instrument has its own particularities, for example, with respect to maturity and degree of liquidity. In this sense, a yield curve model is specified to obtain the nominal interest rate that will define the cost of fixed rate bonds, according to the maturity of the instrument to be issued, as shown in Figure 3.

[25] Nominal exchange rate is obtained by aggregating the differential between domestic and foreign inflation into real exchange rate.

Figure 3. Fixed-rate Curve



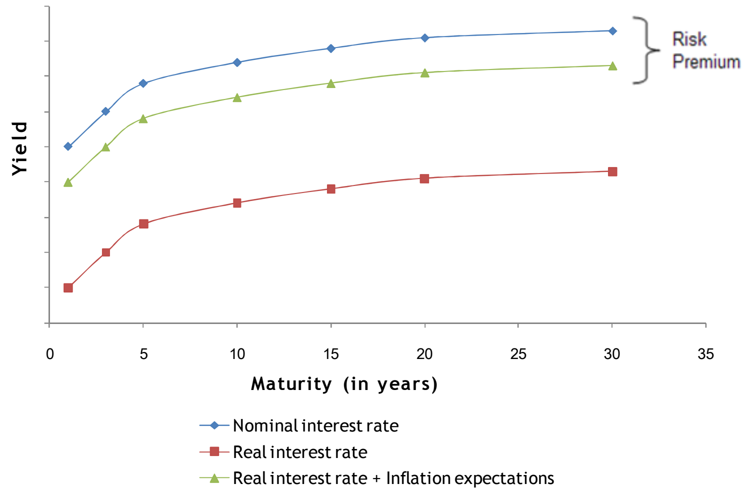
Source: National Treasury

In the Brazilian case, aside from fixed rate bonds, one should also specify yield curves for inflation-linked bonds and for those with linked to exchange rate variations (in the case of the external debt). The model is specified in such a way that the cost of these alternatives is related to the cost of fixed rate bonds with equivalent maturities, adjusted by a risk premium that will be explained below.

The risk premium structure of the model reflects how lower the returns on a security linked to inflation or exchange rate variations should be, compared to bonds with nominal fixed rate yield and equivalent maturities. The idea here is that the existence of a factor that protects real returns on the bond should be provided to the issuer through a lesser risk premium.

Figure 4 illustrates how yield curves are related in the model. The expected cost of inflation-linked securities is composed of the sum of the real interest rate - given by the specific yield curve used to price these securities - plus inflation expectations. This expected cost will be less than the average cost of fixed rate bonds in the presence of a positive inflationary risk premium, since the real returns of the holder of the indexed security are protected against unexpected interest rate variations.

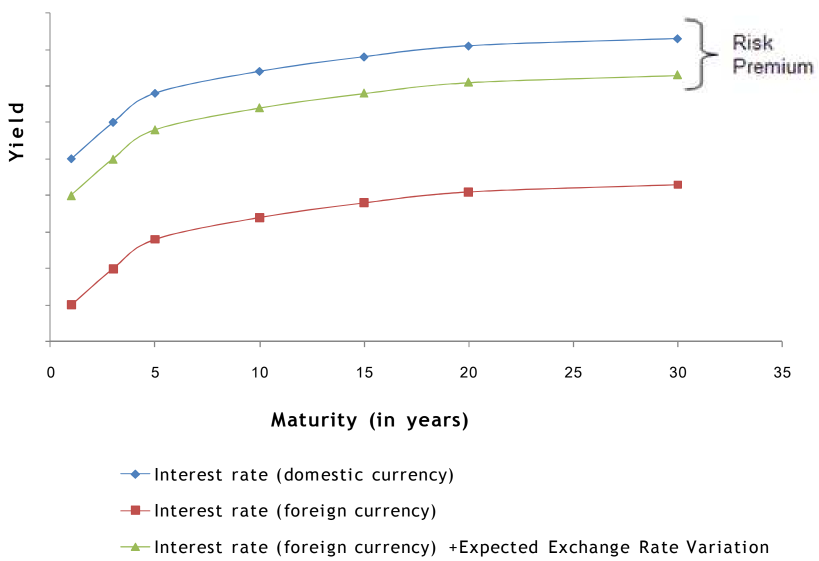
Figure 4. Inflation Risk Premium



Source: National Treasury

In a similar manner, as shown in Figure 5, there is also a direct relation between the yield curve in foreign currency and the curve for fixed rate securities. With respect to bonds denominated in foreign currency, the reason for the risk premium originates in the fact that the external investor desires a hedge against exchange rate fluctuations.

Figure 5. Exchange Risk Premium



Source: National Treasury

Following simulation of the scenarios for financial and macroeconomic variables, the cost of FPD financing is calculated, which depends on the issuance cost of each public bond plus variation of its indexing factor, when appropriate. Appendix 6.4 explains how this calculation is done.

2.4. Dynamics of the Debt

Once the scenarios have been defined, the following accounting identity is used as the starting point for deriving the dynamics of the NPSD and its relation to the FPD.

$$(1) \qquad D_t = X_t + M_t - F_t + L_t^{flutuante} + L_t^{IP} + L_t^{FX}$$

In this identity, the assets and liabilities included in NPSD (D) are grouped into four categories: FPD (X), monetary base (M), international reserves (F) and other net public sector liabilities[26] [indexed to floating rates ($L_t^{flutuante}$), inflation (L_t^{IP}), and exchange rate variation (L_t^{FX})].

Outstanding FPD in period t is equal to the outstanding volume in the previous period plus its carrying cost (c_t), less the primary fiscal result, less variation of the monetary base. Consequently, FPD evolves according to the following equation:

$$(2) \qquad X_t = X_{t-1}(1 + c_t) - S_t - \Delta M_t$$

With respect to the other components of NPSD, it is presumed that the monetary base remains constant as a proportion of GDP over time, while international reserves (equation 3)[27], as well as other net public sector liabilities (equations 4-6) start from an initial volume and evolve according to their returns.

$$(3) \qquad F_t = F_{t-1}(1 + r_t^{\text{reserves}})(1 + \Delta c\hat{a}mbio)$$

$$(4) \qquad L_t^{flutuante} = L_{t-1}^{floating}(1 + \text{Selic}_t)$$

$$(5) \qquad L_t^{IP} = L_{t-1}^{IP}(1 + c_t^{IP})$$

$$(6) \qquad L_t^{FX} = L_{t-1}^{FX}(1 + c_t^{FX})$$

[26] FPD encompasses only federal government liabilities. To shift to the NPSD concept, one must consider that this debt includes nonfinancial public sector liabilities plus the Central Bank. Therefore, this concept includes the direct federal, state and municipal administrations, indirect administrations, the public social security system, nonfinancial government companies, as well as the Central Bank of Brazil. Additionally, as a net concept, NPSD deducts public sector financial assets (for example, international reserves, funds such as the Worker Support Fund and credits with financial institutions) from liabilities. Finally, intragovernmental debts (crossed relations) are excluded, in such a way that only the public sector debt with private agents is measured.

[27] The rate of return on international reserves ($r_t^{reserves}$)in the model may be different from the average cost of the external debt.

In which c^{IP} and c^{FX} are the carrying cost of the bonds indexed to inflation and denominated in foreign currency; and $r_t^{reserves}$) represents the rate of return of international reserves.

After substituting (2) – (6) in (1) and dividing the new equation by GDP, algebraic manipulations lead to the following formula to describe the trajectory of the NPSD/GDP ratio over time:

$$(7) \qquad d_t = x_{t-1}\frac{(1+c_t)}{(1+\gamma_t)} - s_t + \frac{m}{(1+\gamma_t)} - f_{t-1}\frac{(1+c_t^{\text{reserves}})}{(1+\gamma_t)} + l_{t-1}\frac{(1+c_t^l)}{(1+\gamma_t)}$$

In which:

$$c_t^{\text{reserves}} = (1 + r_t^{\text{reserves}})(1 + \Delta câmbio) - 1$$

$$l_t = l_t^{floating} + l_t^{IP} + l_t^{FX}$$

$$c_t^l = \left(Selic_t l_t^{floating} + c_t^{IP} l_t^{IP} + c_t^{FX} l_t^{FX}\right)/l_t$$

3. Optimal Composition Model Simulations

By means of an exercise, this section will demonstrate the application of the model described above to FPD. The following data and results are merely illustrative. In actual practice, aside from the simulations based on a fundamental set of parameters, the robustness of the conclusions is tested when analyzing the sensitivity of the model to variations in benchmark parameters.

At the same time, though the definition an optimal composition (benchmark) is based on inputs derived from simulations, it will depend on a wide-ranging discussion that gives due consideration to the feasibility of adopting a specific debt profile in a given temporal horizon, as well as an understanding of the interactions between debt management and such other economic policies as fiscal and monetary policy.

The first step in the simulation is to obtain the parameters and initial values[28] for the stochastic models and, with this, generate the macroeconomic scenarios. These parameters depend on the stochastic model adopted but, in general, can be described in terms of the average and volatility of the scenarios for each variable. The following tables show the scenarios generated by the models for the macroeconomic variables[29]:

[28] Since the model presupposes work in the stationary state, all of the macroeconomic variables remain in the range of their long-term averages. For this reason, the initial values are the same as the long-term values.

[29] The stochastic processes currently employed by the National Treasury are detailed in Appendix 6.1.

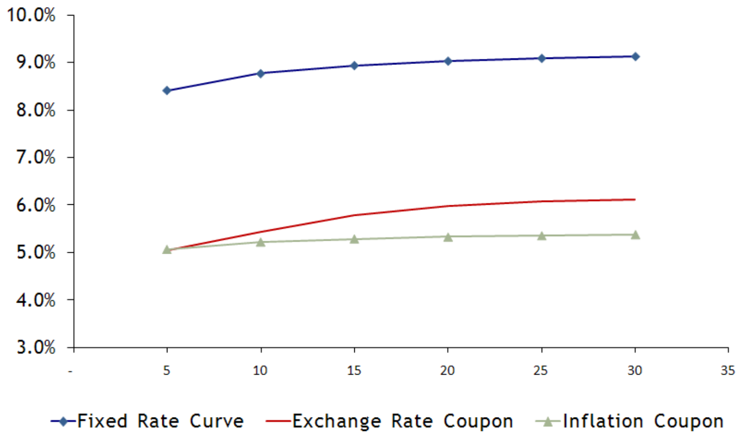
Table 2. Scenarios Generated (cumulative in 12 months)

Per Year	SELIC	IGP-M	CPI	Nominal Exchange Rate	Libor	GDP
Average	7.45%	2.98%	2.99%	2.84%	4.69%	4.50%
Standard Deviation	1.80%	1.11%	0.70%	14.28%	1.52%	2.33%
Percentile 5°	4.79%	1.16%	1.84%	-18.94%	2.51%	0.72%
Percentile 95°	10.67%	4.83%	4.15%	22.77%	7.48%	8.39%

Source: National Treasury

Calculation of the cost of FPD financing also depends on simulation of yield curves. In average terms, the figure below shows the cost of fixed rate bonds, the real yield curve (for inflation-linked bonds) and the yield curve in dollars (for bonds denominated in foreign currency). As already described, once the prices and, consequently, the cost of the fixed rate bonds have been defined, the cost of the inflation-linked bonds is that of the fixed rate bonds less inflation expectations, less an inflation risk premium. Analogously, the cost of exchange rate-indexed bonds is that of fixed rate bonds, less expectations of currency devaluation, less an exchange risk premium.

Figure 6. Fixed Rate Curve, Inflation Coupon and Exchange Rate Coupon



Source: National Treasury]

Table 3 shows the carrying cost of each financing operation for the selected instruments. In this case, aside from the rate of return indicated by the yield curve, the cost is calculated by adding in the variation of the indexing factor of the bond throughout the period. Considering a 10-year horizon, dispersion of the cost is also explained by the dynamics of debt refinancing during the simulations.

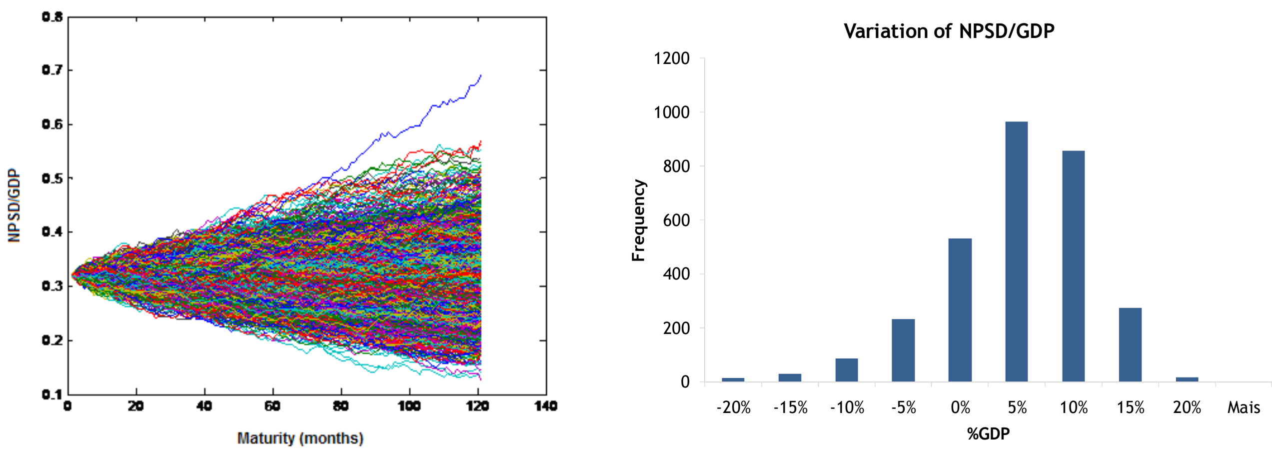
Table3. Financing Cost in 10 Years (cumulative in 12 months)

Per Year	Average	Standard Deviation	Percentile 5	Percentile 95
Fixed 1 year	7.77%	1.40%	5.55%	10.43%
Fixed 3 years	8.14%	0.84%	6.78%	9.75%
Fixed 5 years	8.40%	0.52%	7.55%	9.39%
Fixed 10 years	8.76%	0.16%	8.50%	9.06%
Inflation 10 years	8.21%	2.41%	4.07%	12.50%
Inflation 30 years	8.37%	2.40%	4.24%	12.66%
Exchange rate 10 years	8.24%	68.80%	-56.29%	151.80%
Exchange rate30 years	9.24%	68.91%	-55.91%	154.26%
Selic 5 years	7.51%	1.83%	4.63%	10.99%

Source: National Treasury

Once the costs of each instrument are obtained, the dynamics of the FPD and, after that, of the NPSD, are calculated for portfolios composed 100% of a specific instrument and for portfolios with pairs of securities in a proportion of 50%-50%. The figure below shows the trajectories of 3000 simulations of NPSD/GDP for a portfolio composed 100% of bonds linked to the Selic rate, together with a histogram with the distribution of the variation of the NPSD/GDP in these simulations.

Figure 7. Dynamics of the NPSD/GDP

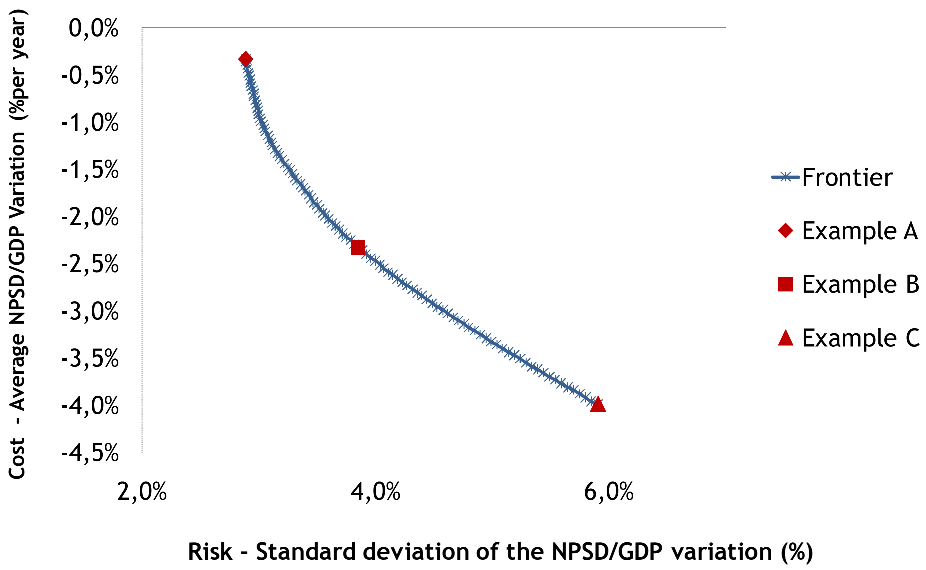


Source: National Treasury

In a manner similar to that presented above, for each simulated portfolio there are the distribution of the NPSD/GDP in the analysis period, its average, the standard deviation and the correlation matrix of the portfolios. Cost is defined as the average of the NPSD/GDP variation and the risk as its standard deviation. With this information, the efficient frontier is generated.

Utilizing the efficient frontier, the results are presented in a graph in the cost/risk space. The frontier may be generated with the nominal cost and risk or with a specific portfolio as reference (for example, the lowest risk composition of the simulations) and performing the calculations related to it. This frontier does not consider any analysis of factibility of the portfolios due to the particularities of the public bond market or other premises for debt management, since it is a frontier with a purely financial bias.

Figure 8. Efficient Frontier



Source: National Treasury

An FPD composition is associated to each point on the efficient frontier above. By way of example, the following Table shows the FPD profile that characterizes points A, B and C highlighted in Figure 8.

Table 4. Composition of the Portfolios

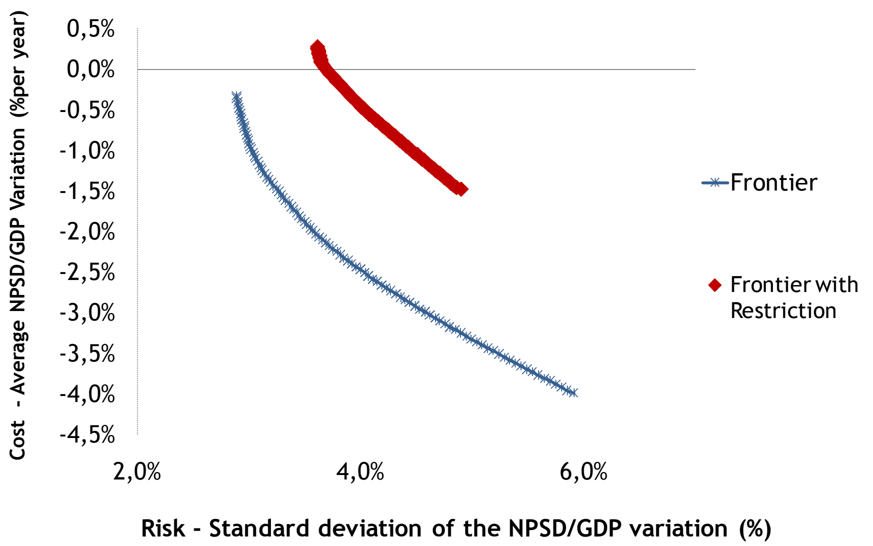
Examples	Portfolio Composition	
	Inflation	Exchange rate
A	100.0%	0.0%
B	50.0%	50.0%
C	0.0%	100.0%

Source: National Treasury

Another interesting characteristic of the model is the possibility of defining technical restrictions on compositions in the efficient frontier. This allows for greater flexibility, since there are some aspects of economic policy or the market that are not directly perceived initially, but that may be defined as restrictions on the efficient portfolio. One should recall that the efficient frontier is a function not only of the composition, but also of the maturity of the instruments. With this, the restrictions may involve any one of these two aspects.

One possible restriction is the existence of a limited demand for a specific type of instrument or, in other words, a limit of feasibility for some FPD compositions. The manager may also define a minimum value for the average maturity of the outstanding volume or a maximum value for the percentage of the debt maturing in 12 months, based on the management guidelines of that debt. The following figure presents a frontier with restrictions, as cited above:

Figure 9. Efficient Frontier with Restrictions

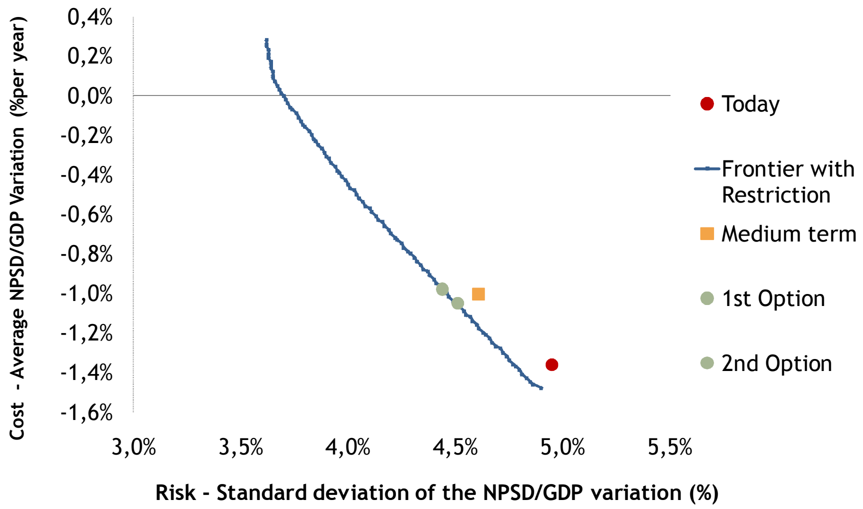


Source: National Treasury

The choice of a composition as benchmark for the public debt implies a choice of the risk that the government (and society) is willing to run and the cost that it is willing to pay in order to protect itself. Given a risk level, the composition of minimum cost may be obtained from the frontier. The debate based on results is wide-ranging and the choice of an optimal composition is viewed in the framework of strategic planning of public debt management.

In order to illustrate the latter point, the following figure shows the efficient frontier with restrictions and, given the parameters utilized, how a debt composition positions itself in relation to the frontier. In this case, one notes that there would be space to increase efficiency in debt management through alterations in the FPD composition toward some portfolio from the frontier.

Figure 10. Efficient Frontier and Medium-Term*



* Note: The references presented in this graph are merely illustrative and, therefore, do not necessarily correspond to the National Treasury's choice of options.
Source: National Treasury

Analysis of the path to be followed toward a specific optimal portfolio must be carried out within the framework of medium-term strategies, seeking to design a transition plan, with due consideration of current debt composition and its maturity structure and how fast convergence to the desired debt profile in the future could be. As an example, Figure 10 shows that the portfolio resulting from the transition strategy (medium-term) would represent a higher cost and lower risk than the current portfolio, while also positioning itself relatively closer to the efficient frontier.

4. Final Considerations

In the case of the Brazilian Federal Public Debt, the initial optimal composition proposal was published in the 2007 Annual Borrowing Plan - ABP[30]. These simulations of the model suggested that efficient management of FPD would be that which leads to an increased proportion of fixed rate securities and inflation-linked securities, in detriment to floating rate and exchange rate-indexed debt. More recently, refinement of studies on definition of the optimal composition (benchmark) based on FPD management objectives and evaluation of risks, restrictions and opportunities in coming years, led to definition of the desired composition, which is presented in the 2011 ABP in the form of indicative long-term limits, as shown in the following table:

Table 5. Optimal Long-Term FPD Composition

	Lower limit	Upper limit
Fixed rate	40%	50%
Inflation-linked	30%	35%
Floating rate	10%	20%
Exchange rate	5%	10%

Source: National Treasury

As stressed in the 2011 ABP, the prescription for seeking the composition described above should be qualified. In the first place, it must be viewed as a guideline to be achieved gradually, without generating pressures that result in excessive transition costs. Secondly, FPD composition must not be sought in a manner uncoordinated with its maturity structure. Lengthening of the average FPD maturity and, more specifically, of that of fixed rate bonds is a necessary condition for the composition suggested for FPD to result in efficiency gains and risk reductions (this result is also derived from the simulations).

In the third place, the cost of the composition change must be permanently monitored. Significant oscillations in the relative prices of the different instruments included in these strategies, due above all to changes in the risk premiums, may result in adjustments in the benchmark portfolio for FPD.

Finally, though these limits provide a current guide for defining strategies, they must also reflect possible restrictions related to the development stage of financial markets in Brazil, to the investor base profile and to the demand and future liquidity outlook for public bonds. The speed of convergence from the current FPD composition to that indicated in Table 5 will depend on overcoming some of these restrictions.

[30] The annual borrowing plans can be found at http://www.tesouro.fazenda.gov.br/english/public_debt/annual_borrowing_plan.asp

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6. Appendix

6.1. Dynamics of the Economy

The interest rate (SELIC and LIBOR) follow a CIR model. In other words, the interest rate process (neutral to risk) is described by:

$$dJ_t = \alpha(J^* - J_t)dt + \sigma_1\sqrt{J_t}dz_t^1$$

In which:

J_t: Spot interest rate (SELIC or LIBOR) at instant t

α: mean reversion rate

J^*: interest rate long-run average

σ_1: interest rate volatility

dz_t^1: Wiener process

The CIR model shows the property of mean reversion, in which parameters "α" and "J^*"

are the mean reversion rate and the long-run average, respectively. Given α>0, whenever the spot rate is below the long-term average, $J_t < J^*$, the trend term is positive, α (J^* - J_t)>0, thus causing the current rate to increase. In much the same way, if the spot rate is above the long-term average, $J_t > J^*$, the trend term becomes negative, α (J^* - J_t) < 0, producing a reduction in the current rate. Such a reversion to the mean would seem to be a highly reasonable hypothesis and a desirable property in interest rate models.

By introducing the term $J_t^{1/2}$ into the volatility component, the CIR model eliminates the possibility of negative short-term interest rates.

The price of bonds (and, consequently, the yield curve) can be derived from the CIR model. More specifically, the price of fixed rate bonds is given by:

$$P(t,T) = A(t,T)e^{-B(t,T)J}$$

in which:

$$B(t,T) = \frac{2\left(e^{\gamma(T-t)} - 1\right)}{(\gamma + \alpha)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma}$$

$$A(t,T)=\left[\frac{2\gamma e^{(\alpha+\gamma)(T-t)/2}}{(\gamma+\alpha)\left(e^{\gamma(T-t)}-1\right)+2\gamma}\right]^{2\alpha J^{*}{}_{1}/\sigma_1^2}$$

$$\gamma=\sqrt{\alpha^2+2\sigma_1^2}$$

However, one should observe that the CIR model does not precisely represent market prices and, therefore, does not precisely represent the Brazilian economy. But this is not our objective. Since it is aimed at generating stationary state scenarios, with interest rates at their equilibrium level, the CIR model is strongly intuitive since it has an explicit formula for bond prices and is based on an equilibrium model. Other more precise pricing models, such as HJM do not have these advantages and, as forecasting models, are just as arbitrary as CIR.

The CKLS model[31] was adopted for the real exchange rate, with the rate exponent in the volatility term equal to one, in which the process would be described by:

$$dC_t=\beta(C^*-C_t)dt+\sigma_2 C_t dz_t^2$$

In which:

C_t : spot real exchange rate

β : mean reversion rate

C^* : exchange rate long-run average

σ_2: real exchange rate volatility

dz_t^2 : Wiener process

The CKLS model is a generalization of the CIR model that can be written as $dr_t=a(b-r_t)dt+\sigma r_t^{\gamma}dz$. Therefore, we are using a model of this type for the real rate of exchange, with $\gamma=1$.

The real exchange rate equation can be rewritten as:

$$\frac{dC_t}{C_t}=\beta\left(\frac{C^*}{C_t}-1\right)dt+\sigma_2 dz_t^2$$

We observe that C^*/C_t is nothing more than the deviation (in fact, the inverse of the deviation) of the real rate of exchange in relation to its equilibrium value. In this case, when the exchange rate is devalued, in other words, $C^*/C_t<1$, we have $E[dC_t/C_t]<0$, or rather, real devaluation of the exchange rate is expected. In summary, the process adopted for real exchange shows mean reversion, in

[31] The CKLS model is a general function that encompasses the CIR model.

such a way that the volatility component of the exchange variation rate does not depend on the exchange rate level.

In the case of the price index, a Brownian geometric process was defined:

$$dI_t = \mu I_t dt + \sigma_3 I_t dz_t^3$$

In which:

I_t : spot price index

μ : average rate of growth of price index

σ_3 : volatility of price index

dz_t^3 : Wiener process

The Wiener process used in the three models is a Markov process. If “z” follows a Wiener process and a small change in “z” is observed, Δz, in a small time interval Δt, then the diffusion process of “z” has the following properties: (i) $\Delta z = \varepsilon\sqrt{\Delta t}$, in which $\varepsilon \approx N(0,1)$ and (ii) for two distinct intervals Δt, the corresponding Δz are independent (Markov).

A variable “x” follows a generalized Wiener process, if its diffusion process can be written as:

$$dx = adt + bdz$$

In which “z” follows a Wiener process, as described above. This generalized Wiener process involves the idea of a tendency component (adt) and another of volatility (bdz).

The Itō process is an extension of the generalized Wiener process in which the coefficients of the tendency and volatility components are variables, and may depend on the value of the variable itself, the process of which is being described, and on time. Consequently, an Itō process can be described as:

$$dx = a(x,t)dt + b(x,t)dz$$

Therefore, the process that we are adopting for the price index is specifically an Ito process:

$$dI_t = \mu I_t dt + \sigma_3 I_t dz_t^3$$

Since it is widely utilized, this process receives a special name of geometric Brownian motion. It can also be written as:

$$\frac{dI_t}{I_t} = \mu dt + \sigma_3 dz$$

Therefore, the rate of inflation variation has a constant trend component. Aside from this, this rate for any small time interval is normally distributed and, given two distinct small time intervals, the variation rates are distinct. We can write that:

$$\frac{dI_t}{I_t} \approx N(\mu, \sigma_3)$$

A deterministic process referring to the external price index was also used and is given by:

$$dI_t^e = \mu^e I_t^e dt$$

In which:

I_t^e : external price index at instant t

μ^e : growth rate of external price index.

One knows that the nominal exchange rate N_t can be calculated as:

$$N_t = \frac{I}{I_t^e} C_t$$

Applying the Itō lemma in the latter equation, the nominal exchange process is defined as:

$$dN_t = N_t \left[\frac{dI_t}{I_t} + \frac{dC_t}{C_t} - \frac{dI_t^e}{I_t^e} + 2\rho_{23}\sigma_2\sigma_3 dt \right]$$

in which:

ρ_{23} : Correlation coefficient between the real exchange rate and the domestic inflation rate

Finally, starting with the diffusion processes of real exchange rate and domestic and external price indices, one obtains the process for the nominal exchange rate;

$$\frac{dN_t}{N_t} = \left[\beta \left(\frac{I_t}{I_t^e} \frac{C^*}{N_t} - 1 \right) + \mu - \mu_e + 2\rho_{23}\sigma_2\sigma_3 \right] dt + \sigma_2 dz_t^2 + \sigma_3 dz_t^3$$

The real GDP scenario is also stochastic and is modeled by a geometric Brownian motion.

$$\frac{dPIB_t}{PIB_t} = \mu_{PIB} dt + \sigma_4 dz$$

Therefore, the rate of GDP variation has a constant trend component (μ_{PIB}). Aside from this, this rate for any small time interval is normally distributed in such a way that:

$$\frac{dPIB_t}{PIB_t} \approx N(\mu_{PIB}, \sigma_4)$$

In conclusion, the nominal GDP is the combination of the real GDP with the inflation scenario.

6.2. Macroeconomic Consistency

Each one of the three primitive processes modeled (interest rate, real exchange rate and inflation) has a random term characterized by the Wiener process. However, one knows that there is a correlation among the macroeconomic variables. For example, it is difficult to imagine a situation in which interest rate drops at the same time in which inflation rises. Here, the Cholesky factorization method is used to create correlated random numbers in order to attain macroeconomic consistency for the model.

Consider an estimated covariance matrix for the macroeconomic variable **S**. Since the covariance matrix is always symmetrical and semi-defined positive, it can be decomposed $\mathbf{S} = \mathbf{CC}^T$, in which **C** is an inferior triangular matrix. Consider $\boldsymbol{\omega}_t$ a vector with *n* random variables with normal distribution and $\mathbf{dz}_t$ the vector of correlated random numbers. $\mathbf{dz}_t = \mathbf{C}\,\boldsymbol{\omega}_t$ is defined. Thus:

$$\mathbf{cov(dz_t) = E\left(dz_t dz_t^T\right) = E\left(C\omega_t\omega_t^T C^T\right) = CIC^T = CC^T = S}$$

Thus, the covariance matrix is generated for the Wiener process, in such a way that one ensures macroeconomic consistency in the correlation among the variables.

The interest rate, real exchange rate and inflation scenarios are correlated, even if the variables distance themselves temporarily from equilibrium, precisely as occurs historically. Parallel to this, an appropriate correlation structure can bring macroeconomic consistency to the model, without the need for generating arbitrary theoretical relations among the variables. With this, we have a model in which the variables always return to their equilibrium value, since small deviations from equilibrium are permitted.

6.3. Bond Prices

The carrying cost of the debt depends on the issuance cost of each instrument. As already demonstrated, the LTN price (fixed rate bond) is defined by the CIR model, based on the following equation:

$$P(t,T) = A(t,T)e^{-B(t,T)J}$$

Since LTN are fixed rate bonds, their carrying cost is obviously the rate at which they are issued.

With regard to LFT (Selic rate bonds), the model assumes they have been sold at par or, in other words, that their price is equal to face value. Consequently, their cost is defined by the daily compound Selic interest rate during the period.

The prices of inflation and exchange-linked bonds are a function of the fixed rate with the same maturity adjusted by a risk premium. This premium represents how much the rate of these bonds must be below the fixed rate with the same maturity . In other words, each premium indicates the reduction applied to the fixed rate, in such a way as to obtain the external issuance rate or the real issuance rate of domestic bonds indexed to inflation.

These premiums are modeled by the Nelson-Siegel equation, which associates a premium (P) and a term (T), given the parameters β_0, β_1, β_2 and κ, according to the formula below:

$$P = \beta_0 + (\beta_1 + \beta_2 \cdot T) \cdot e^{-kT}$$

In the case of inflation-linked securities, the justification for the existence of a premium that fixed rate bonds pay over and above the NTN-B is based in the fact that investors who acquire these bonds are protected against variations in the inflation. As far as foreign debt bonds are concerned, investors are protected against exchange rate volatility.

The effect of the premium modeled by the Nelson-Siegel equation can be better understood graphically. As shown in the following graph, the real yield curve is defined as the fixed rate issuance rate, less the risk premium, less inflation expectations for the period. In the same manner, the external issuance rate is the domestic fixed rate, less the risk premium, less expectations of exchange rate devaluation

6.4. Carrying Cost

The carrying cost of each group of bonds depends on their characteristics of return: the fixed rate in the case of LTN; the SELIC rate in the case of LFT; the real interest rate plus inflation, in the case of NTN-B; and exchange rate coupon plus exchange variation, in the case of external debt securities.

Since it was assumed that LFT are sold at par, their carrying cost is simply the Selic rate for the period:

$$R_t^{LFT} = J_t$$

The carrying cost of the LTN in each period is the weighted average of the issuance costs of all outstanding LTN, as shown in the equation below:

$$R_t^{LTN} = \sum_{s=0}^{n} \omega_{t-s} r_{t-s}$$

In which each ω_{t-s} is a percentage in t of the fixed rate debt issued in t-s; and r_{t-s} is the issuance cost of LTN in t-s.

For bonds indexed to exchange rate, the carrying cost is composed of a combination of nominal exchange rate evolution and the coupon, weighted by the outstanding bonds. This average cost R_t^C is calculated in a similar manner:

$$R_t^C = \sum_{s=0}^{n} \omega_{t-s}^c r_{t-s}^c$$

In which each ω_{t-s}^{C} is a percentage of the outstanding exchange rate-linked bonds at moment t that were issued in t-s; and r_{t-s}^{c} is the interest coupon issued at moment t-s.

Adding in exchange rate variation (dN/N), the total cost of the exchange debt is given by:

$$R_t^{FX} = \left(1+\frac{dN_t}{N_t}\right)(1+R_t^C)-1$$

The cost of inflation-linked bonds (NTN-B) is calculated in a manner similar to the cost of exchange rate-indexed bonds. The real interest rate for each period, R_t^I, is also calculated as the average of all the real rates of outstanding bonds:

$$R_t^I = \sum_{s=0}^{n} \omega_{t-s}^{i} r_{t-s}^{i}$$

In which ω_{t-s}^{i} is a percentage in t of the outstanding inflation-linked bonds in t-s; and r_{t-s}^{i} is the real interest rate of a bond issued in t-s. Considering correction for inflation (dI/I) aside from the real rate, the carrying cost of NTN-B is given by:

$$R_t^{NTN-B} = \left(1+\frac{dI_t}{I_t}\right)\left(1+R_t^I\right)-1$$

For any composition of FPD, the total cost is given by the weighted average of the carrying cost of each instrument. This cost is necessary for calculating the cost of FPD and, consequently, for the dynamics of NPSD:

$$R_t^D = \lambda_{LFT} R^{LFT} + \lambda_{LTN} R^{LTN} + \lambda_{FX} R^{FX} + \lambda_{NTN-B} R^{NTN-B}$$

In which R_t^D is the carrying cost of the portfolio; and $\lambda_{LFT}, \lambda_{LTN}, \lambda_{NTN-B}, \lambda_{FX}$ represents the participation of each type of bond in the chosen composition of the portfolio.